APEC 8002: Mathematical review

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Identities

- $x^{a+b} = x^a x^b$
- $(x^a)^b = x^{ab}$
- $(xy)^a = x^a y^a$
- $x^2 - y^2 = (x+y)(x-y)$
- $\log(xy) = \log x + \log y$
- $\log\left(\frac{x}{y}\right) = \log x - \log y$
- $\log(x^a) = a \log x$
- $x_1 + x_2 + \cdots + x_N = \sum_{n=1}^{N} x_n$
- $x_1 x_2 \ldots x_N = \prod_{n=1}^{N} x_n$

Total differentiation

- Given a function $f(x, y)$, its total differential is

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

- Given a function $f(x, y, w)$, its total differential is

$$df = f_x' dx + f_y' dy + f_w' dw$$

Additional note:

(Euler's Theorem[1]) Suppose $f: \mathbb{R}_{++}^n \to \mathbb{R}$ and it is homogeneous of degree α. Then,

$$\sum_{n=1}^{N} \frac{\partial f}{\partial x_n} x_n = \alpha f(x) \qquad \forall x \in \mathbb{R}_{++}^n$$

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[1] **Simplified proof** for the curious ones: Assume f is homogeneous of degree α. Then, $f(\beta x) = \beta^\alpha f(x), \forall \beta > 0$.
Totally differentiate w.r.t. β on both sides (assume $x' = \beta x$)

$$\frac{\partial f}{\partial x'} \frac{\partial x'}{\partial \beta} = \alpha \beta^{\alpha-1} f(x)$$

$$\frac{\partial f}{\partial x'} x = \alpha f(x) \qquad \because let\ \beta = 1$$

This can be proved for other β, but that's for even more curious people.

Exercises

For the following exercise, please work with one partner. I will assign a random number to your group. Solve the exercises enumerated with the number I give you. Then, please write the solution on the board.

A. Write the short-form of the following series (Hint: find expressions that have either Σ or Π operators).Example: $1+1^2+\cdots+1^N=\sum_{n=0}^{N}1^n=\sum_{n=0}^{N}1=N$

1. $ax_1+ax_2+\cdots+ax_N-y_1-y_2-\cdots-y_N=\sum_{n=1}^{N}(ax_n-y_n)$
2. $x_0+x_1(y-a)+x_2(y-a)^2+\cdots+x_N(y-a)^N=\sum_{n=0}^{N}x_n(y-a)^n$
3. $\ln(x_1\,x_2\ldots x_N)=\ln(\Pi_{n=1}^{N}x_n)=\sum_{n=1}^{N}\ln(x_n)$
4. $\ln(x_1^a x_2^a\ldots x_N^a)=\ln(\Pi_{n=1}^{N}x_n^a)=\sum_{n=1}^{N}\ln(x_n^a)=a\sum_{n=1}^{N}\ln(x_n)$
5. $\frac{1}{x^{a_1}}\frac{1}{x^{a_2}}\ldots\frac{1}{x^{a_n}}=\frac{1}{\ln(\Pi_{n=1}^{N}x_n^a)}=\frac{1}{a\sum_{n=1}^{N}\ln(x_n)}$
6. $\ln\left(\frac{1}{x^{a_1}}\frac{1}{x^{a_2}}\ldots\frac{1}{x^{a_n}}\right)=\ln\left(\frac{1}{a\sum_{n=1}^{N}\ln(x_n)}\right)=-a\sum_{n=1}^{N}\ln(x_n)$

B. Expand the following series. Example: $\prod_{n=1}^{N}x_n=x_1x_2\ldots x_N$

7. $\prod_{n=1}^{N}\ln(e^a)=\Pi_{n=1}^{N}a=aN$
8. $\sum_{n=1}^{N}(x_{n+1}-x_n)=(x_2-x_1)+(x_3-x_2)+\cdots+x_{N+1}-x_N=x_{N+1}-x_1$
9. $\ln(\prod_{n=1}^{N}x_n)=\sum_{n=1}^{N}\ln(x_n)=\ln(x_1)+\ln(x_2)+\cdots+\ln(x_N)$
10. $\prod_{n=1}^{N}e^{a_n\ln(x_n)}=\prod_{n=1}^{N}e^{\ln(x_n^{a_n})}=\prod_{n=1}^{N}x_n^{a_n}=x_1^{a_1}x_2^{a_2}\ldots x_N^{a_N}$
11. $\frac{\prod_{n=1}^{N}(x_n^2-4)}{\prod_{n=1}^{N}(x_n-2)}=\frac{\prod_{n=1}^{N}(x_n-2)\prod_{n=1}^{N}(x_n+2)}{\prod_{n=1}^{N}(x_n-2)}=\prod_{n=1}^{N}(x_n+2)$

C. Calculate the total differential of the following functions

1. $f(x,y)=\frac{y}{\prod_{n=1}^{N}\ln(e^x)}=\frac{y}{x^N}$

$$df=-\frac{Ny}{x^{N+1}}dx+\frac{1}{x^N}dy$$

2. $f(x,y)=e^{\ln x^2}-3(xy)^{-2}$

$$df=(2x+6x^{-3}y^{-2})dx+6x^{-2}y^{-3}dy$$

3. $f(x,y)=\frac{e^x}{x}y$

$$df=e^xy\left(\frac{1}{x}-\frac{1}{x^2}\right)dx+\frac{e^x}{x}dy$$

4. $f(x,y)=x\sum_{n=1}^{N}y=xy^N$

$$df=y^Ndx+Nxy^{N-1}dy$$

5. $f(x,y)=\frac{\sum_{n=1}^{N}x}{x}y=\frac{x^Ny}{x}=x^{N-1}y$

$$df=(N-1)x^{N-2}ydx+x^{N-1}dy$$

6. $f(x,y)=5^x+3x^7+2^y$

$$df = [5^x \ln(5) + 21x^6]dx + 2^y \ln 2\, dy$$

7. $f(x,y) = \frac{\sqrt{x}}{x^{1/2}-1} + y$

$$df = \left[\frac{1}{2x^{0.5}(x^{0.5}-1)} - \frac{1}{2(x^{0.5}-1)^2}\right] dx + dy$$

8. $f(x,y) = x^3y - x^2y^2$

$$df = [3x^2y - 2xy^2]dx + [x^3 - 2x^2y]dy$$

9. $f(x,y) = \ln x^2 + \ln y^3 = 2\ln(x) + 3\ln(y)$

$$df = \frac{2}{x}dx + \frac{3}{y}dy$$

10. $f(x,y) = (x-y)(x+y) = x^2 - y^2$

$$df = 2xdx - 2ydy$$

11. $f(x,y) = x^2 + xy + y$

$$df = [2x + y]dx + [x + 1]dy$$