Problem Set 3 - Answer Key

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1. Imagine that you are reading a paper in which the author uses the indirect utility function $v(p_1, p_2, w) = w/p_1 + w/p_2$. You suspect that the author's conclusions in the paper are the outcome of the 'fact' that the function v is inconsistent with the rational behaviour as defined in class. Take the following steps to check whether this is the case:

   (a) Use Roy's identity to derive the demand function.
   Sol.

$$x_1 = -\frac{\partial v/\partial p_1}{\partial v/\partial w} = \frac{w}{p_1}\frac{p_2}{p_1 + p_2}$$
$$x_2 = -\frac{\partial v/\partial p_2}{\partial v/\partial w} = \frac{w}{p_2}\frac{p_1}{p_1 + p_2}$$

   (b) Show that if demand is derived from any smooth utility function (continuous and differentiable), then the indifference curve at the point (x_1, x_2) has the slope $-\sqrt{x_2}/\sqrt{x_1}$.
   Sol.

$$\frac{x_1}{x_2} = \frac{p_2^2}{p_1^2}$$
$$-\sqrt{\frac{x_1}{x_2}} = -\frac{p_2}{p_1}$$

   (c) Construct a utility function with the property that the ratio of the partial derivatives at the bundle (x_1, x_2) is $-\sqrt{x_2}/\sqrt{x_1}$.
   Sol.
   There are many options but one could be $u(x_1, x_2) = 2x_1^{1/2} + 2x_2^{1/2}$

   (d) Calculate the indirect utility function derived from this utility function. Do you arrive at the original $v(p_1, p_2, w)$? If not, can the original indirect utility function still be derived from another utility function satisfying the property in (c)?
   Sol.
   As we already know the solution is interior, then I can skip the Kuhn Tucker conditions.

$$\frac{MU_1}{MU_2} = \frac{p_1}{p_2} \Rightarrow x_2 = \left(\frac{p_1}{p_2}\right)^2 x_1$$

Plugging that in the B.C. You'll find that:

$$x_1 = \frac{p_2 w}{p_1(p_1+p_2)} \quad \text{and} \quad x_2 = \frac{p_1 w}{p_2(p_1+p_2)}$$

Then, the indirect utility is:

$$v(p_1,p_2,w) = 2\sqrt{\frac{p_2 w}{p_1(p_1+p_2)}} + 2\sqrt{\frac{p_1 w}{p_2(p_1+p_2)}}$$

which is not the same as the utility given by the question. However, try using the following utility function, just for fun!: $u(x_1,x_2) = \frac{1}{4}\left(2x_1^{1/2} + 2x_2^{1/2}\right)^2$. You will find the original indirect utility function.

2. Consider the utility function $u(x_1,x_2) = \frac{1}{2}x_1^2 + \ln x_2$.

(a) Are these preferences convex?

Sol.

No. One way to check this is using the second derivative of x_2 with respect to x_1. First solve for x_2 (it should end up in terms of x_1 and u), and then do the following:

$$\frac{\partial^2 x_2}{\partial x_1^2} = exp\left(u - \frac{1}{2}x_1^2\right)(x_1^2 - 1)$$

This is positive if $x_1 > 1$ but negative if $x_1 < 1$, so we have non convex preferences when $x_1 < 1$

(b) Find the Marshallian demand at $p_1 = 2, p_2 = 1$.

Sol.

The lagrangean for this problem is:

$$\mathcal{L} = \frac{1}{2}x_1^2 + \ln x_2 + \lambda[w - 2x_1 - x_2]$$

F.O.C.

$$\frac{\partial \mathcal{L}}{\partial x_1} = x_1 - 2\lambda \le 0; \qquad \frac{\partial \mathcal{L}}{\partial x_1}x_1 = 0$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{1}{x_2} - \lambda \le 0; \qquad \frac{\partial \mathcal{L}}{\partial x_2}x_2 = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = w - 2x_1 - x_2 = 0 \text{ because Walras' Law should hold}$$

i. $x_1 > 0, x_2 > 0$. From this case, we find that there are two for x_1 (and therefore, for x_2)

$$x_1 = \frac{w \pm \sqrt{w^2 - 16}}{4} \text{ and } x_2 = \frac{w \pm \sqrt{w^2 - 16}}{2}$$

However we can discard one root. Plugging both roots in the utility function, we obtain that the biggest utility corresponds to the following demand functions:

$$x_1 = \frac{w + \sqrt{w^2 - 16}}{4} \text{ and } x_2 = \frac{w - \sqrt{w^2 - 16}}{2} \text{ if } w \geq 4$$

ii. $x_1 = 0, x_2 > 0$. In this case the marshallian demands are $x_1 = 0$ and $x_2 = w$. This case is an optimum whenever $w < 4$ (Again, check the indirect utility function).

iii. $x_1 > 0, x_2 = 0$. This case is not possible because of the shape of the utility function (Log of 0 is undefined).

iv. $x_1 = 0, x_2 = 0$. Not possible because of the same reason as the third case.

Therefore, the Walrasian demands are:

$$\begin{cases} x_1 = \dfrac{w + \sqrt{w^2 - 16}}{4} \text{ and } x_2 = \dfrac{w \pm \sqrt{w^2 - 16}}{2} \text{ if } w \geq 4 \\ x_1 = 0 \text{ and } x_2 = w \text{ if } w < 4 \end{cases}$$

3. True or false: if no good is a strict luxury good, then no good can have wealth elasticity of 0.6.

Sol.
False. Let $\mathcal{E}_w = \dfrac{\partial x}{\partial w}\dfrac{w}{x}$

- If $\mathcal{E}_w > 0$, then x is a normal good
  - If $\mathcal{E}_w > 1$, then x is a luxury good
  - If $\mathcal{E}_w \in (0, 1)$, then x is a necessity good
- If $\mathcal{E}_w > 0$, then x is an inferior good

4. Consider the following utility functions:

(a) $U(x, y) = (x + 2y - \dfrac{y^2}{2})$ for $x \geq 0$ and $y \geq 0$. Let $w > 0$ be the wealth the consumer has available to spend on x and y. Let $p_x > 0$ and $p_y > 0$ be the competitive prices for x and y. For case (a), assume $\dfrac{w}{p_x} \geq 1$ and $\dfrac{p_y}{p_x} + \dfrac{w}{p_y} > 2$.

i. Derive the consumer's Walrasian demands.

$$\begin{cases} x(p, w) = \dfrac{w}{p_x} + \dfrac{p_y^2}{p_x^2} - 2\dfrac{p_y}{p_x} \text{ and } y = 2 - \dfrac{p_y}{p_x} \text{ if } 2p_x > p_y \\ x = \dfrac{w}{p_x} \text{ and } y = 0 \text{ if } 2p_x \leq p_y \end{cases}$$

ii. Derive the consumer's indirect utility function.

$$V(p,w) = \begin{cases} \dfrac{w}{p_x} + \dfrac{1}{2}\left(2 - \dfrac{p_y}{p_x}\right)^2 \text{ if } 2p_x > p_y \\ \dfrac{w}{p_x} \text{ if } 2p_x \le p_y \end{cases}$$

iii. Verify that the indirect utility function is: 1. homogeneous of degree 0 in p and w; 2. strictly increasing in w and non-increasing in p; 3. quasiconvex in p only for the utility function in case (a).

1. $V(p,w)$ is HD0 in (p,w)

$$V(\alpha p, \alpha w) = \begin{cases} \dfrac{\alpha w}{\alpha p_x} + \dfrac{1}{2}\left(2 - \dfrac{\alpha p_y}{\alpha p_x}\right)^2 \text{ if } 2p_x > p_y \\ \dfrac{\alpha w}{\alpha p_x} \text{ if } 2p_x \le p_y \end{cases}$$

Notice that $V(\alpha p, \alpha w) = V(p,w)$, then it is HD0.

2. Strictly increasing in w and non-increasing in p

$$\frac{\partial V(p,w)}{\partial w} = \frac{1}{p_x} > 0 \quad \because \quad p_x > 0$$

$$\frac{\partial V(p,w)}{\partial p_x} = \begin{cases} -\dfrac{w}{p_x^2} + \left(2 - \dfrac{p_y}{p_x}\right)\dfrac{p_y}{p_x^2} < 0 \quad \because \quad \dfrac{p_y}{p_x} + \dfrac{w}{p_y} > 2 \\ -\dfrac{w}{p_x^2} < 0 \end{cases}$$

$$\frac{\partial V(p,w)}{\partial p_y} = \begin{cases} -\left(2 - \dfrac{p_y}{p_x}\right)\dfrac{1}{p_x} < 0 \text{ if } 2p_x > p_y \\ 0 \quad \text{if} \quad 2p_x \le p_y \end{cases}$$

3. Quasiconvex in p

$$\frac{\partial^2 V(p,w)}{\partial p_x^2} = \begin{cases} 2\dfrac{w}{p_x^3} - 4\dfrac{p_y}{p_x^3} + 3\dfrac{p_y^2}{p_x} > 0 \quad \text{if } 2p_x > p_y \quad \because \quad 3\dfrac{p_y}{p_x} + 2\dfrac{w}{p_y} > 4 \\ 2\dfrac{w}{p_x^3} > 0 \end{cases}$$

$$\frac{\partial^2 V(p,w)}{\partial p_y^2} = \begin{cases} \dfrac{1}{p_x^2} > 0 \text{ if } 2p_x > p_y \\ 0 \quad \text{if} \quad 2p_x \le p_y \end{cases}$$

iv. Show whether Roy's Law satisfied?

Yes. You can easily show that by using Roy's law: $x_l(p,w) = -\dfrac{\partial V/\partial p_l}{\partial V/\partial w}$, you obtain:

$$\begin{cases} x(p,w) = \dfrac{w}{p_x} + \dfrac{p_y^2}{p_x^2} - 2\dfrac{p_y}{p_x} \text{ and } y = 2 - \dfrac{p_y}{p_x} \text{ if } 2p_x > p_y \\ x = \dfrac{w}{p_x} \text{ and } y = 0 \text{ if } 2p_x \leq p_y \end{cases}$$

v. Derive the expenditure function.

To obtain the expenditure function we solve for w in the indirect utility function. That is:

$$e(p,u) = \begin{cases} up_x - \dfrac{p_x}{2}\left(2 - \dfrac{p_y}{p_x}\right)^2 \text{ if } 2p_x > p_y \\ up_x \text{ if } 2p_x \leq p_y \end{cases}$$

vi. Derive the Hicksian demands using Shephard's Lemma.

$$\begin{cases} h_x = u - 2 + \dfrac{1}{2}\left(\dfrac{p_y}{p_x}\right)^2 \text{ and } h_y = 2 - \dfrac{p_y}{p_x} \text{ if } 2p_x > p_y \\ h_x = u \text{ and } h_y = 0 \text{ if } 2p_x \leq p_y \end{cases}$$

(b) $U(x,y) = \ln((x^{-1} + y^{-1})^{-1})$ for $x > 0$ and $y > 0$

i. Derive the consumer's Walrasian demands.

$$\begin{cases} x(p,w) = \dfrac{w}{p_x + \sqrt{p_x p_y}} \\ y(p,w) = \dfrac{w}{p_y + \sqrt{p_x p_y}} \end{cases}$$

ii. Derive the consumer's indirect utility function.

$V(p,w) = \ln\left(\dfrac{w}{p_x + 2\sqrt{p_x p_y}}\right)$

iii. Verify that the indirect utility function is: 1. homogeneous of degree 0 in p and w; 2. strictly increasing in w and non-increasing in p; 3. quasiconvex in p only for the utility function in case (a).

1. $V(p,w)$ is HD0 in (p,w)

$$V(\alpha p, \alpha w) = \ln\left(\frac{\alpha w}{\alpha p_x + 2\sqrt{\alpha p_x \alpha p_y}}\right) = V(w,p)$$

2. Strictly increasing in w and non-increasing in p

$$\frac{\partial V(p,w)}{\partial w} = \frac{w}{p_x + p_y + 2\sqrt{p_x p_y}} \frac{1}{[\sqrt{p_x} + \sqrt{p_y}]^2}$$

$$\frac{\partial V(p,w)}{\partial p_x} = -\frac{w}{p_x + p_y + 2\sqrt{p_x p_y}} \frac{w}{[\sqrt{p_x} + \sqrt{p_y}]} \frac{1}{\sqrt{p_x}}$$

$$\frac{\partial V(p,w)}{\partial p_y} = -\frac{w}{p_x + p_y + 2\sqrt{p_x p_y}} \frac{w}{[\sqrt{p_x} + \sqrt{p_y}]} \frac{1}{\sqrt{p_y}}$$

iv. Show whether Roy's Law satisfied?

Yes. You can easily show that by using Roy's law: $x_l(p,w) = -\dfrac{\partial V/\partial p_l}{\partial V/\partial w}$, you obtain: the same result as in (b.i)

v. Derive the expenditure function.

Solving for w in the indirect utility function we obtain:
$e(p,u) = [p_x + p_y + 2\sqrt{p_x p_y}]exp\{u\}$

vi. Derive the Hicksian demands using Shephard's Lemma .

$$\begin{cases} h_x(p,u) = e^u \left[1 + \sqrt{\dfrac{p_y}{p_x}}\right] \\ h_y(p,u) = e^u \left[1 + \sqrt{\dfrac{p_x}{p_y}}\right] \end{cases}$$