Problem Set 5 - Answer Key

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1. An individual has w eggs and two baskets. One basket just keeps the eggs safe. The eggs in the other basket might increase and multiply, or they might not. Each egg in this basket yields Z eggs (the increase being $Z-1$), where Z is a random variable that is uniformly distributed on the interval $\left[\frac{2}{3}, \bar{a}\right]$. The individual maximizes expected utility, with Bernoulli utility function is $u(c) = log(a+c)$, where c is the number of eggs consumed, and $a \geq 0$. Will this person put all of the eggs in one basket? Answer this question separately for each of the two following cases:

   (a) If $\bar{a} = \frac{4}{3}$ (can use a dominance theorem)

   Yes. Here, each egg in the 2nd basket has final value z; where z is drawn from a uniform distribution on $\left[\frac{2}{3}, \frac{4}{3}\right]$. So, the expected value of one egg in the second basket is 1, which is the same as the certain value if each egg goes into the first basket. Thus, even if he splits up the eggs between the first and second baskets, he is getting a mean-preserving spread of putting all his eggs in the safe basket. A risk-averse individual dislikes mean-preserving spreads. So, since this individual is risk-averse, he wants everything in the first basket.

   (b) If $\bar{a} = \frac{5}{3}$

   Suppose this person puts x eggs in the risky basket and the remaining $(w-x)$ in the safe basket. Then, his final wealth will be $w - x + xz = w + (z-1)x$. Then the expected utility is:

   $E[U] = \int_{2/3}^{5/3} \ln(a + w + (z-1)x)dz$

   The derivative with respect to x is

   $\frac{\partial E[U]}{\partial x} = \int_{2/3}^{5/3} \frac{z-1}{a+w+(z-1)x}dz$

   - If he wants to put all eggs in the first basket, then $x = 0$. So:
     $\frac{\partial E[U]}{\partial x} = \int_{2/3}^{5/3} \frac{z-1}{a+w}dz = \frac{1}{a+w}\left(\frac{z^2}{2} - z\right)\Big|_{2/3}^{5/3} > 0$. Then this is not optimal.

- If he wants to put all eggs in the second basket, then $x = w$. So:
$\frac{\partial E[U]}{\partial x} = \int_{2/3}^{5/3} \frac{z-1}{a+wz} dz = \int_{2/3}^{5/3} \left[\frac{1}{w} - \left(\frac{a}{w} + 1 \right) \frac{1}{a+wz} \right] dz$
$= \left[\frac{1}{w} z - \left(\frac{a}{w} + 1 \right) \ln(a+wz) \frac{1}{w} \right] \Big|_{2/3}^{5/3}$
$= \frac{1}{w} - \frac{a+w}{w^2} \ln\left(\frac{3a+5w}{3a+2w} \right)$
$= \frac{1}{w} \left[1 - \frac{a+w}{w} \ln\left(\frac{3a+5w}{3a+2w} \right) \right]$
We have to check whether $\frac{a+w}{w} \ln\left(\frac{3a+5w}{3a+2w} \right)$ is smaller or equal to 1. Let's define $y = \frac{a}{w}$.
Then, rewriting this expression:

$$\left(\frac{a}{w} + 1 \right) \ln \left(\frac{3\frac{a}{w} + 5}{3\frac{a}{w} + 2} \right) = (1+y) \ln \left(\frac{3y+5}{3y+2} \right) = f(y)$$
$$f(y) = \frac{\ln(3y+5) - \ln(3y+2)}{\frac{1}{1+y}}$$

  - If $y \to 0$, then :

    $\lim_{y \to 0} f(y) = \ln(5) - ln(2) = 0.91 < 1$

  - If $y \to \infty$, then (using l'Hopital):
    $\lim_{y \to \infty} f(y) = \lim_{y \to \infty} \frac{\frac{3}{3y+5} - \frac{3}{3y+2}}{-\frac{1}{(1+y)^2}} = \lim_{y \to \infty} \frac{9(1+y)^2}{(3y+5)(3y+1)} = 1$

In other words, that $f(y)$ is always less than 1, thus $\frac{\partial E[U]}{\partial x} > 0$. Therefore, it is not optimal to put all eggs in the same basket.

2. Consider the St. Petersburg Paradox (*check https://plato.stanford.edu/entries/paradox-stpetersburg/*) and a lottery that pays 2^n dollars with a chance 2^{-n} for all $n = 1, 2, ...$ Which of the following utility functions would solve this Paradox?

   (a) $u(x) = x^\alpha$
   $E[U] = \sum_{n=1}^{\infty} 2^{-n} (2^n)^\alpha = \sum_{n=1}^{\infty} (2^{\alpha - 1})^n$

   This is finite only if $\alpha \leq 1$

   (b) $u(x) = \ln x$

   $E[U] = \sum_{n=1}^{\infty} 2^{-n} \ln(2^n) = \sum_{n=1}^{\infty} \frac{n \ln(2)}{2^n} = ln(2) \sum_{n=1}^{\infty} \frac{n}{2^n}$

   This converges if the following expression has a limit:

$$\lim_{n\to\infty} \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}$$

That means that this utility solves the Paradox.

3. Consider the following two distributions over returns:

$$G(x) = x^2, \text{ on } [0,1]$$
$$F(x) = \frac{x-a}{b-a} \text{ on } [a,b]$$

Under what conditions is G preferred to F by anyone who likes money?
Notice first that $G(x)$ and $F(x)$ must always be between zero and one (because they are cumulative distribution functions). Thus, it is true that $x \in [0,b]$. Remember that in this model we don't have debts, reason why, the lower bound is zero. $G(x)$ dominates $F(x)$ if:

$$\int_0^1 x^2 dx - \int_a^b \frac{x-a}{b-a} dx \leq 0$$

This leads to our first condition: $b \geq \frac{2}{3} + a$. The second condition comes from the fact that it must also be true that $G(x)$ must dominate $F(x)$ at the extremes (i.e. at $x=0$ and $x=b$).

If $x=0$, then $0^2 - \frac{0-a}{b-a} \leq 0$. Then, the second condition is that $a \leq 0$.
If $x=b$, then $b^2 - \frac{b-a}{b-a} \leq 0$. Then, the third condition is that $b \leq 1$.

Also notice that we need a fourth condition that is trivial. It comes from the fact that the denominator of $F(x)$ shouldn't be zero. Then: $b \neq a$

4. An agent is a strictly risk-averse with initial wealth $w > 100$. He risks losing 100 in an accident (chance p). An insurance company offers him insurance at a rate r: That is, he can pay rx and insure for x of loss, so that is the accident happens, the insurance company reimburses x. Suppose that $p < r < 2p$. True or False: He always buys partial insurance (i.e., $x \in (0, 100)$). Explain your answer.

He wants to maximize:

$$\max_x pu(w - 100 + x - rx) + (1-p)u(w - rx)$$

F.O.C.

$$p(1-r)u'(w - 100 + x - rx) - r(1-p)u'(w - rx)$$

To choose full insurance, this would need to be ≥ 0 at $x = 100$, i.e.,

$$p(1-r)u'(w-100r) \geq r(1-p)u'(w-100r)$$
$$p \geq r$$

So according to the initial conditions, he does not buy full insurance. Notice that if this person wasnt to buy zero insurance, then the derivative must be ≤ 0 at $x = 0$

5. True or False: Constant relative risk aversion implies that the demand for insurance is a decreasing function of wealth. Explain your answer.

Assume that the utility function is $u(x) = \dfrac{x^{1-\rho}}{1-\rho}$ (You can prove that this is CRRA). Let's prove it using the example given in the last question: A person has a potential loss of L (instead of 100) with probability p. The price per dollar of insurance coverage is r. We know that $u'(x) = x^{-\rho}$. So, the F.O.C at the optimum becomes:

$$p(1-r)(w-100+x-rx)^{-\rho} - r(1-p)(w-rx)^{-\rho} = 0$$
$$\frac{(w-100+x-rx)^{-\rho}}{(w-rx)^{-\rho}} = \frac{r(1-p)}{p(1-r)}$$

Let $z = \dfrac{r(1-p)}{p(1-r)}$, and solve for x. You should arrive to something like:

$$x = \frac{w(z^{-1/\rho}-1)+L}{rz^{-1/\rho}+1-r}$$

So x is a decreasing function of wealth iff $z^{-1/\rho} < 1$, which implies $r > p$