Recitation 5: Preliminary exam solutions

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Summer 2017. Question II.2.

a) For the state contingent output q_a and q_b, what are the producer's conditional input demands?

$$z_1(\mathbf{p},\mathbf{r},\mathbf{R}) = \frac{\partial C(\mathbf{r},\mathbf{p},\mathbf{R})}{\partial r_1} = r_1^{-\frac{1}{2}} r_2^{\frac{1}{2}}(q_a^2 + q_b^2)$$
$$z_2(\mathbf{p},\mathbf{r},\mathbf{R}) = \frac{\partial C(\mathbf{r},\mathbf{p},\mathbf{R})}{\partial r_2} = r_1^{\frac{1}{2}} r_2^{-\frac{1}{2}}(q_a^2 + q_b^2)$$

b) Suppose a risk averse producer's utility of profit function is

$$W(\pi_a, \pi_b) = 3 - e^{-\mu\pi_a} - 2e^{-\mu\pi_b}$$

where $\mu > 0$,

i) What is the producer's subjective probabilities for state a and b?

Recall that

$$\phi_s = \frac{\frac{\partial W(1^S)}{\partial \pi_S}}{\sum_{s'=1}^{S} \frac{\partial W(1^S)}{\pi_{s'}}}$$

Notice that:

$$\frac{\partial W(\pi_a,\pi_b)}{\partial \pi_a} = \mu e^{-\mu\pi_a} \quad \rightarrow \quad \frac{\partial W(1^S)}{\partial \pi_a} = \mu e^{-\mu}$$
$$\frac{\partial W(\pi_a,\pi_b)}{\partial \pi_b} = 2\mu e^{-\mu\pi_b} \quad \rightarrow \quad \frac{\partial W(1^S)}{\partial \pi_b} = 2\mu e^{-\mu}$$

Therefore,

$$\phi_a = \frac{\mu e^{-\mu}}{\mu e^{-\mu} + 2\mu e^{-\mu}} = \frac{1}{3}$$
$$\phi_a = \frac{2\mu e^{-\mu}}{\mu e^{-\mu} + 2\mu e^{-\mu}} = \frac{2}{3}$$

ii) For π_a and π_b in general, what is the producer's certainty equivalent profit?

Recall that

$$ce(\boldsymbol{\pi}) = \min_c\{c \in \mathbb{R}: W(c\mathbf{1}^S) \geq W(\boldsymbol{\pi})\}$$

Thus,

$$3 - e^{-\mu c} - 2e^{-\mu c} \geq 3 - e^{-\mu\pi_a} - 2e^{-\mu\pi_b}$$
$$3e^{-\mu c} \leq e^{-\mu\pi_a} + 2e^{-\mu\pi_b}$$
$$c \geq -\frac{1}{\mu}\ln\left(\frac{e^{-\mu\pi_a} + 2e^{-\mu\pi_b}}{3}\right)$$
$$ce(\boldsymbol{\pi}) = -\frac{1}{\mu}\ln\left(\frac{1}{3}e^{-\mu\pi_a} + \frac{2}{3}e^{-\mu\pi_b}\right)$$

Does the certainty equivalent profit increase or decrease as μ increases?

$$\frac{\partial ce(\boldsymbol{\pi})}{\partial \mu} = \frac{1}{\mu^2}\underbrace{\ln\left(\frac{1}{3}e^{-\mu\pi_a} + \frac{2}{3}e^{-\mu\pi_b}\right)}_{\leq 0} + \frac{1}{\mu}\underbrace{\frac{\pi_a e^{-\mu\pi_a} + 2\pi_b e^{-\mu\pi_b}}{e^{-\mu\pi_a} + 2e^{-\mu\pi_b}}}_{\geq 0} > 0$$

What does this result imply about the relationship between a producer's risk aversion and μ?

Recall that $arp(\boldsymbol{\phi}, \boldsymbol{\pi}) = \bar{\pi} - ce(\boldsymbol{\pi})$. Then, if μ increases, the coefficient of risk aversion will decrease through its effect on the $ce(\boldsymbol{\pi})$. The person becomes less risk averse.

c) Suppose instead that the producer is risk neutral with subjective probability beliefs $\phi_a > 0$ and $\phi_b > 0$ for state a and b.

i) Find this producer's optimal outputs for states a and b. You can assume the solution is interior

$$\max_{q_a, q_b} E\boldsymbol{\pi} = \beta\left[p_a q_a - 2r_1^{\frac{1}{2}}r_2^{\frac{1}{2}}(q_a^2 + q_b^2)\right] + (1-\beta)\left[p_b q_b - 2r_1^{\frac{1}{2}}r_2^{\frac{1}{2}}(q_a^2 + q_b^2)\right]$$
$$= \beta p_a q_a + (1-\beta)p_b q_b - 2r_1^{\frac{1}{2}}r_2^{\frac{1}{2}}(q_a^2 + q_b^2)$$

$F.O.C.$

$$\frac{\partial E\pi}{\partial q_a} = \beta p_a - 4r_1^{\frac{1}{2}}r_2^{\frac{1}{2}}q_a \leq 0 \quad ; \frac{\partial E\pi}{\partial q_a}q_a^* = 0 \quad ; q_a^* \geq 0$$
$$\frac{\partial E\pi}{\partial q_b} = (1-\beta)p_b - 4r_1^{\frac{1}{2}}r_2^{\frac{1}{2}}q_b \leq 0 \quad ; \frac{\partial E\pi}{\partial q_b}q_b^* = 0 \quad ; q_b^* \geq 0$$

Assuming interior solution, then the F.O.C. are binding. Therefore,

$$q_a^{RN} = \frac{\beta p_a}{4r_1^{\frac{1}{2}}r_2^{\frac{1}{2}}}$$
$$q_b^{RN} = \frac{(1-\beta)p_b}{4r_1^{\frac{1}{2}}r_2^{\frac{1}{2}}}$$

ii) Given these optimal outputs, what are the producer's unconditional input demands?

Using the results in a)

$$z_1 = r_1^{-\frac{1}{2}}r_2^{\frac{1}{2}}(q_a^2 + q_b^2) = r_1^{-\frac{1}{2}}r_2^{\frac{1}{2}}\left(\frac{\beta^2 p_a^2}{16r_1 r_2} + \frac{(1-\beta)^2 p_b^2}{16r_1 r_2}\right)$$
$$z_2 = r_1^{\frac{1}{2}}r_2^{-\frac{1}{2}}(q_a^2 + q_b^2) = r_1^{\frac{1}{2}}r_2^{-\frac{1}{2}}\left(\frac{\beta^2 p_a^2}{16r_1 r_2} + \frac{(1-\beta)^2 p_b^2}{16r_1 r_2}\right)$$

Spring 2017. Question II.2.

a) Suppose $W(\pi_g, \pi_b) = \alpha\pi_g + \sqrt{\pi_g\pi_b} + \beta\pi_b$ is the risk-averse producer's utility of profit

(i) What are its subjective probabilities for the good and bad states?

$$\frac{\partial W(\pi)}{\partial \pi_g} = \alpha - \frac{1}{2}\pi_g^{-0.5}\pi_b^{0.5} \quad \rightarrow \quad \frac{\partial W(1^S)}{\partial \pi_g} = \alpha - \frac{1}{2}$$
$$\frac{\partial W(\pi)}{\partial \pi_b} = \beta - \frac{1}{2}\pi_g^{0.5}\pi_b^{-0.5} \quad \rightarrow \quad \frac{\partial W(1^S)}{\partial \pi_g} = \beta - \frac{1}{2}$$
$$\phi_g = \frac{\alpha - 0.5}{\alpha + \beta - 1}$$
$$\phi_b = \frac{\beta - 0.5}{\alpha + \beta - 1}$$

(ii) For π_g and π_b in general, what is the producer's absolute risk premium given these preferences?

Step 1: Find the $ce(\pi)$

$$ce(\pi) = \min_c\{c \in \mathbb{R}: W(c\mathbf{1}^S) \geq W(\boldsymbol{\pi})\}$$
$$\alpha c + c + \beta c \geq \alpha\pi_g + \sqrt{\pi_g\pi_b} + \beta\pi_b$$
$$c \geq \frac{\alpha\pi_g + \sqrt{\pi_g\pi_b} + \beta\pi_b}{\alpha + \beta + 1}$$
$$ce(\pi) = \frac{\alpha\pi_g + \sqrt{\pi_g\pi_b} + \beta\pi_b}{\alpha + \beta + 1}$$

Step 2: Find $arp(\phi, \pi)$

$$arp(\phi, \pi) = \bar{\pi} - ce(\pi)$$
$$arp(\phi, \pi) = \bar{\pi} - \frac{\alpha\pi_g + \sqrt{\pi_g\pi_b} + \beta\pi_b}{\alpha + \beta + 1}$$
$$= \frac{(\phi_g\pi_g + \phi_b\pi_b)(\alpha + \beta + 1) - \alpha\pi_g - \sqrt{\pi_g\pi_b} - \beta\pi_b}{\alpha + \beta + 1}$$

b) Suppose instead that there are two types of producers. The first maximizes its expected profit where $\phi_g > \phi_b > 0$ are its subjective probabilities such that $\phi_g + \phi_b = 1$. The second maximizes the general utility of profit function $W(\pi_a, \pi_b)$ where $\frac{\partial W(\pi_a,\pi_b)}{\partial \pi_a} > 0$ and $\frac{\partial W(\pi_a,\pi_b)}{\partial \pi_b} > 0$. Furthermore, assume this producer is risk averse such that this utility function is generalized Schur-concave with respect to the probabilities ϕ_g and ϕ_b

i) Set up the profit-maximizing producer's optimization problem and derive the first order condition for the interior solution.

$$\max E\boldsymbol{\pi} = \phi_g\left(\mu(z) + \frac{h(z)}{\phi_g} - rz\right) + \phi_b\left(\mu(z) - \frac{h(z)}{\phi_b} - rz\right) = \mu(z) - rz$$

F.O.C.

$$\frac{\partial E\pi}{\partial z} = \mu'(z) - r \leq 0; \qquad \frac{\partial E\pi}{\partial z}z^* = 0; \qquad z^* \geq 0$$

The condition at the optimum is then $\mu'(z^{RN}) = r$

ii) Set up the utility-maximizing producer's optimization problem and derive the first order condition for the interior solution

$$\max EW(\boldsymbol{\pi})$$

F.O.C.

Recall that

$$W(\pi_g, \pi_b) = \alpha\pi_g + \sqrt{\pi_g \pi_b} + \beta\pi_b$$
$$\pi_g = \mu(z) + \frac{h(z)}{\phi_g} - rz$$
$$\pi_b = \mu(z) - \frac{h(z)}{\phi_b} - rz$$

$$\frac{\partial EW(\pi)}{z} = \frac{\partial W}{\partial \pi_g}\frac{\partial \pi_g}{\partial z} + \frac{\partial W}{\partial \pi_b}\frac{\partial \pi_b}{\partial z}$$

Let $W_s = \frac{\partial W}{\partial \pi_s}, \forall s = g, b$

$$\frac{\partial EW(\pi)}{z} = W_g\left(\mu'(z) + \frac{h'(z)}{\phi_g} - r\right) + W_b\left(\mu'(z) - \frac{h'(z)}{\phi_b} - r\right) \leq 0$$
$$(W_g + W_b)(\mu'(z) - r) + W_g\frac{h'(z)}{\phi_g} - W_b\frac{h'(z)}{\phi_b} \leq 0$$
$$\mu'(z) + \frac{W_g}{W_g + W_b}\frac{h'(z)}{\phi_g} - \frac{W_b}{W_g + W_b}\frac{h'(z)}{\phi_b} \leq r$$

The condition at the optimum is then

$$\mu'(z^{RA}) + \frac{W_g}{W_g + W_b}\frac{h'(z^{RA})}{\phi_g} - \frac{W_b}{W_g + W_b}\frac{h'(z^{RA})}{\phi_b} = r$$

Notice that $\phi_s = \frac{W_s}{W_g + W_b}, \forall s = g, b$. Therefore,

iii) Assuming $\mu'(z) > 0$ and $\mu''(z) < 0$, what conditions on $h'(z)$ are required for the risk-averse farmer to use less of the input than the expected profit maximizing farmer? Justify your answer and explain the intuition of your result.

We have to find the conditions for which $z^{RA} < z^{RN}$. Given the shape of the $\mu(z)$ function, this implies $\mu'(z^{RA}) > \mu'(z^{RN})$. In turn, this implies:

$$r - \frac{W_g}{W_g + W_b}\frac{h'(z^{RA})}{\phi_g} + \frac{W_b}{W_g + W_b}\frac{h'(z^{RA})}{\phi_b} > r$$
$$\frac{W_b}{W_g + W_b}\frac{h'(z^{RA})}{\phi_b} > \frac{W_g}{W_g + W_b}\frac{h'(z^{RA})}{\phi_g}$$

$$\left(\frac{W_g}{\phi_g} - \frac{W_b}{\phi_b}\right) < 0$$

From the generalized Shur-concave preferences, we know

$$\left(\frac{W_g}{\phi_g} - \frac{W_b}{\phi_b}\right)\left(\pi_g - \pi_b\right) \leq 0$$

For it to hold, then $\left(\pi_g - \pi_b\right) \geq 0$

$$\mu(z) + \frac{h(z)}{\phi_g} - rz - \mu(z) + \frac{h(z)}{\phi_b} + rz \geq 0$$

$$\frac{h(z)}{\phi_g} + \frac{h(z)}{\phi_b} \geq 0$$

This is always true. Therefore, the risk averse producer always uses less input than the risk neutral producer. No additional condition on h is required.